

Creating Quadratic Functions

- In our previous lessons, we have looked at sequences where the equation was given to us. Now, we are going to look at problems where we will determine the equation of a quadratic sequence by solving a system of equations. In Math 11, we used three points to find the equation of a quadratic function of the form $y = ax^2 + bx + c$. We needed to use three points because we needed to solve for three variables. Now, we are working with a quadratic sequence of the form $t_n = an^2 + bn + c$. By using the fact that $D_2 = 2a$, we can find the value of the variable “a” by calculating the value of D_2 and rearranging the formula to be $a = \frac{D_2}{2}$. Now that we have the value for “a”, we only need to use two points to find the values of the variables “b” and “c”.
- Example 1:

- (a.) Create a quadratic function $t_n = an^2 + bn + c$ to generate the sequence $\{0, 10, 26, 48, 76, 110, \dots\}$.
- (b.) Use the function to determine the 26th term in the sequence.

Solution:

(a.)

Step 1: Determine the value of D_2 .

		$\{0, 10, 26, 48, 76, 110, \dots\}$				
First level difference		V	V	V	V	V
Between terms (D_1)	→	10	16	22	28	34
Second level difference		V	V	V	V	
Between terms (D_2)	→	6	6	6	6	

Since there is a common difference at D_2 , the given sequence is quadratic where $D_2 = 6$.

So, $a = \frac{D_2}{2}$
 $a = \frac{6}{2}$
 $a = 3$

Therefore, we have the quadratic function $t_n = 3n^2 + bn + c$.

Step 2: Select any two values from the sequence and substitute into the quadratic function. This is possibly more easily done if we look at the sequence as a table of values:

n	1	2	3	4	5	6
t_n	0	10	26	48	76	110

Therefore, we can use any two points from the table above. We will use the points (1, 0) and (2, 10), where $t_1 = 0$ and $t_2 = 10$.

Step 3: Substitute the points (1, 0) and (2, 10) into the quadratic function $t_n = 3n^2 + bn + c$ to create a system of two quadratic equations.

$$\begin{aligned}
 t_n &= 3n^2 + bn + c \\
 t_1 &= 3(1)^2 + b(1) + c \\
 0 &= 3 + b + c \\
 b + c &= 0 - 3 \\
 b + c &= -3
 \end{aligned}$$

$$\begin{aligned}
 t_n &= 3n^2 + bn + c \\
 t_2 &= 3(2)^2 + b(2) + c \\
 10 &= 12 + 2b + c \\
 2b + c &= 10 - 12 \\
 2b + c &= -2
 \end{aligned}$$

We now have the system of equations: $\begin{cases} b + c = -3 \\ 2b + c = -2 \end{cases}$.

Step 4: Solve the system of equations to determine the values of b and c.

Using substitution, we can solve the system of equations in the following way:

Rearranging Equation 1, we get:

$$b + c = -3$$

$$c = -3 - b$$

Substituting $c = -3 - b$ into Equation 2, we get:

$$2b + (-3 - b) = -2$$

$$2b - 3 - b = -2$$

$$b - 3 = -2$$

$$b = 1$$

Substituting $b = 1$ into Equation 1, we get:

$$1 + c = -3$$

$$c = -4$$

Therefore, the solution to the system of equations is: $b = 1$ and $c = -4$.

Step 5: Substitute the values of b and c into the quadratic function $t_n = 3n^2 + bn + c$.

Replacing the values of b and c , we now have the quadratic function $t_n = 3n^2 + n - 4$.

$$\begin{aligned} \text{(b.) } t_n &= 3n^2 + n - 4 \\ t_{26} &= 3(26)^2 + (26) - 4 \\ t_{26} &= 3(676) + 26 - 4 \\ t_{26} &= 2050 \end{aligned}$$

Therefore, the 26th term of the sequence is 2050.

- Example 2:

(a.) Create a quadratic function $t_n = an^2 + bn + c$ to generate the sequence $\{2, 7, 15, 26, 40, 57, \dots\}$.

(b.) Use the function to determine the 18th term in the sequence.

Solution:

(a.)

Step 1: Determine the value of D_2 .

		{2, 7, 15, 26, 40, 57, ...}				
First level difference		V	V	V	V	V
Between terms (D_1)	→	5	8	11	14	17
Second level difference		V	V	V	V	
Between terms (D_2)	→	3	3	3	3	

Since there is a common difference at D_2 , the given sequence is quadratic where $D_2 = 3$.

So,

$$a = \frac{D_2}{2}$$
$$a = \frac{3}{2}$$

Therefore, we have the quadratic function $t_n = \frac{3}{2}n^2 + bn + c$.

Step 2: Select any two values from the sequence and substitute into the quadratic function. This is possibly more easily done if we look at the sequence as a table of values:

n	1	2	3	4	5	6
t_n	2	7	15	26	40	57

Therefore, we can use any two points from the table above. We will use the points (1, 2) and (2, 7), where $t_1 = 2$ and $t_2 = 7$.

Step 3: Substitute the points (1, 2) and (2, 7) into the quadratic function $t_n = \frac{3}{2}n^2 + bn + c$ to create a system of two quadratic equations.

$$t_n = \frac{3}{2}n^2 + bn + c$$

$$t_1 = \frac{3}{2}(1)^2 + b(1) + c$$

$$2 = \frac{3}{2} + b + c$$

$$b + c = 2 - \frac{3}{2}$$

$$b + c = \frac{1}{2}$$

$$t_n = \frac{3}{2}n^2 + bn + c$$

$$t_2 = \frac{3}{2}(2)^2 + b(2) + c$$

$$7 = \frac{12}{2} + 2b + c$$

$$2b + c = 7 - 6$$

$$2b + c = 1$$

We now have the system of equations: $\begin{cases} b + c = \frac{1}{2} \\ 2b + c = 1 \end{cases}$.

Step 4: Solve the system of equations to determine the values of b and c .

Using substitution, we can solve the system of equations in the following way:

Rearranging Equation 1, we get:

$$b + c = \frac{1}{2}$$

$$c = \frac{1}{2} - b$$

Substituting $c = \frac{1}{2} - b$ into Equation 2, we get:

$$2b + \left(\frac{1}{2} - b\right) = 1$$

$$b + \frac{1}{2} = 1$$

$$b = 1 - \frac{1}{2}$$

$$b = \frac{1}{2}$$

Substituting $b = \frac{1}{2}$ into Equation 1, we get:

$$\begin{aligned}\frac{1}{2} + c &= \frac{1}{2} \\ c &= 0\end{aligned}$$

Therefore, the solution to the system of equations is: $b = \frac{1}{2}$ and $c = 0$.

Step 5: Substitute the values of b and c into the quadratic function $t_n = \frac{3}{2}n^2 + bn + c$.

Replacing the values of b and c , we now have the quadratic function $t_n = \frac{3}{2}n^2 + \frac{1}{2}n$.

$$\begin{aligned}\text{(b.) } t_n &= \frac{3}{2}n^2 + \frac{1}{2}n \\ t_{18} &= \frac{3}{2}(18)^2 + \frac{1}{2}(18) \\ t_{18} &= \frac{3}{2}(324) + 9 \\ t_{18} &= 495\end{aligned}$$

Therefore, the 18th term of the sequence is 495.